The Fredholm Alternative and Its Application

Real Analysis and Functional Analysis

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Part I

The Statement in Linear Algebra

Theorem 1

If V is an n-dimensional vector space and $T : V \to V$ is a linear transformation, then exactly one of the following holds:

- *For each vector* $\mathbf{v}$ *in* V *there is a vector* $\mathbf{u}$ *in* V *so that* $T(\mathbf{u}) = \mathbf{v}$. *In other words: T is surjective (and so also bijective, since V is finite-dimensional).*
- *dim(ker(T))>0.*

Part II

Definitions and the Theorem

Definition 2 (Hilbert Space)

A linear space X over the field of real or complex numbers is called Euclidean if $X \times X$ is equipped with a function (· , ·) with values in the respective field such that:

- $(x, x) \geq 0$ and, in addition, (x,x) = 0 only for x = 0,
- $(x, y) = \overline{(y, x)}$ for all $x, y \in X$ (in the real case: (x,y) = (y,x)),
- $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$ for all $x, y, z \in X$ and all scalars $\alpha,\ \beta$

A function with the stated properties is called an inner product. A complete Euclidean space is called a **Hilbert space**.

Definition 3 (Compact Operator)

Let X and Y be Hilbert spaces. A linear operator $K : X \to Y$ is called **compact** if it takes the unit ball to a set with compact closure.

Theorem 4 (The Fredholm Alternative)

Let K be a compact operator on a complex or real Banach space X. Then

$$Ker(K\text{-}I)=0 \Leftrightarrow (K\text{-}I)(X)=X,$$

i.e., **EITHER** *the equation*

$$Kx\text{-}x = y$$

is uniquely solvable for all $y \in X$*,*
OR *for some vector* $y \in X$ *it has no solutions and then the homogeneous equation*

$$Kx\text{-}x = 0$$

has nonzero solutions.

Part III

Application: The Poisson Equation

Let's consider the Poisson Equation

$$\begin{cases} -\Delta u = f & in\, D \\ u = 0 & on\, \partial D \end{cases}$$

where $u \in W_0^{1,2}(D), f \in \mathcal{L}^2(D)$ and D is an open bounded subset of $\mathbb{R}^n$.

We want to show $f \in \mathcal{L}^2(D)$, $\exists!$ weak solution $u \in W_0^{1,2}(D)$ solves the Poisson equation.

$$\forall \phi \in W_0^{1,2}(D), \text{ we have } \int_D \langle \nabla u, \nabla \phi \rangle = \int_D \phi f$$

Theorem 5 (Lax-Milgram)

Let H be a Hilbert space and $B : H \times H \to H$ a bilinear map s.t.

- *B is bounded, that is, $\exists\, K > 0$ s.t. $|B[x,y]| \leq K\|x\|\|y\|,\ \forall x, y \in H$;*
- *B is coercive, that is, $\exists\, c > 0$ s.t. $B(x,x) \geq c\|x\|^2,\ \forall x \in H$*

If $F \in H^*$ is any bounded linear functional then

$$\exists! w_F \in H s.t. F(x) = B(w_F, x) \forall x \in H$$

According to Lax-Milgram, we have $\forall g \in \mathcal{L}^2(D)$,

$$-\Delta u + u = g$$

has a unique weak solution. Thus we can define

$$(-\Delta + I)^{-1} : \mathcal{L}^2(D) \to W_0^{1,2}(D) \hookrightarrow \mathcal{L}^2(D)$$

SKETCH

STEP 3: COMPACTNESS OF $(-\Delta + I)^{-1}$

- $(-\Delta + I)^{-1} : \mathcal{L}^2(D) \to W_0^{1,2}(D)$ is continuous;
- $W_0^{1,2}(D) \hookrightarrow \mathcal{L}^2(D)$ is compact

Theorem 6 (Rellich–Kondrachov theorem)

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain, and let $1 \leq p < n$. Set

$$p^* = \frac{np}{n-p},$$

Then the Sobolev space $W^{1,p}(\Omega)$ is continuously embedded in the $\mathcal{L}^p$ space $\mathcal{L}^{p^*}(\Omega)$ and is compactly embedded in $\mathcal{L}^q(\Omega)$ for every $1 \leq q < p$.

- $(-\Delta + I)^{-1}$ is compact.

SKETCH

STEP 4: FREDHOLM ALTERNATIVE

STEP 5: REPHRASING

EITHER the equation

$$u - (-\Delta + I)^{-1}u = h$$

is uniquely weakly solvable for all $h \in W_0^{1,2}(D)$,
OR,

$$u - (-\Delta + I)^{-1}u = 0$$

has non-trivial weak solutions.

Let's rephrase the result if we let $f = (-\Delta + I)^{-1}h \in \mathcal{L}^2(D)$,
EITHER the equation

$$-\Delta u = f$$

has unique weak solution for all $f \in \mathcal{L}^2(D)$,
OR,

$$-\Delta u = 0$$

has non-trivial weak solutions.

Step 6: Weyl's Lemma and the Strong Maximum Principle

By Weyl's Lemma and strong maximum principle, we have

$$Ker(-\Delta) = 0$$

Thus the equation

$$-\Delta u = f$$

has unique weak solution for all $f \in \mathcal{L}^2(D)$.

References

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